COMPRESSIVE SURVEY ON ADVANCE INVERSE GALOIS THEORY PROBLEM FOR DEL PEZZO SURFACES FINITE FIELDS

[1*]A VIJAYALAKSHMI,V SUJATHA[2]

[12]ASSISTANT PROFESSOR, DEPARTMENT OF MATHEMATICS, SRI INDU INSTITUTE OF ENGINEERING AND TECHNOLOGY, SHERIGUDA, TELANGANA 501510, INDIA

ABSTRACT.

The overarching subject of the task is Inverse Galois Theory (IGT): understanding certain logarithmic frameworks with a given Galois structure. We totally take care of the inverse Galois issue for Del Pezzo surfaces of degree 2 and 3 over every single finite field. The prospects and procedures may some way or another vary however our arrangement at last is that the joined accomplishments on these different features add to some noteworthy advances, including on the IGP. Another objective is to get a predictable and reviewed vision (even somewhat assumed) of the conveyance of group's w.r.t. to our IGT organized picture. Our IGT issues can be considered over different field's k than Q, consequently actuating new issues, which for some are intriguing for the good of they own and regardless fill in as a direction for the major circumstance over Q. We consider neighborhood, Hilbertian, PAC fields, and despite the fact that we for the most part limit to trademark 0, the positive trademark circumstance is a conceivable heading of our task. Acknowledging finite groups is our fundamental center, yet taking care of installing issues and moving to expert finite groups are common speculations. At last having a modulifide space see is a further improving and continually remunerating methodology.

I. Introduction

In mathematics, Galois Theory gives an association between field theory and gathering theory. Utilizing Galois Theory, certain issues in field theory can be decreased to gather theory, which is in some sense more straightforward and better comprehended.

Galois' work, disregarding its incredible significance, came to be distributed just in 1846, by Joseph Liouville (1809-1882), in the Journal of Mathematiques Pures et Appliquees. In any case, its 60-page content was still difficult to comprehend and did not excite more noteworthy enthusiasm for established researchers at the time. Regardless of the commitment of Enrico Betti (1823-1892) in the feeling of to make this content clearer, specifying certain still dark sections, and finishing a few showings, the thoughts and revelations of Galois were just known and acknowledged in all their am-plitude with the distribution in 1870 by Camille Jordan (1838-1922) of his "Traite des substitutions et des equations algebriques". From that point, the theory of Galois picks up reputation and rouses the emerging of comparative speculations in more broad settings. We notice, by method for representation, a portion of the works distributed in the period 1880-1950, due individually to

I) Charles Emile Picard (1856-1941) and Ernest Vessiot (1865-1952): a differential Galois theory for homogeneous straight differential equations (see [58],

(ii) Wolfgang Krull (1899-1971): a Galois theory for field augmentations of in finite measurement [61],

(iii) Henri Cartan (1904-2008) and Nathan Jacobson (1910-1999): a Galois theory for division rings [20, 52],

(iv) Jean Dieudonn (1906-1992), Gerhard Hochschild (1915-2010), Goro Azu-maya (1920-2010) and Tadasi Nakayama (1912-1964): a Galois theory for straightforward rings [33, 51, 4, 71, 72].

The subject is named after Évariste Galois, who presented it for concentrate the underlying foundations of a polynomial and portraying the polynomial equations that are resolvable by radicals as far as properties of the change gathering of their foundations—an equation is feasible by radicals if its foundations might be communicated by a formula including just whole numbers, nth roots and the four fundamental number-crunching activities.

The theory has been promoted (among mathematicians) and created by Richard Dedekind, Leopold Kronecker and Emil Artin, and others, who, specifically, translated the stage gathering of the roots as the automorphism gathering of a field augmentation.

Galois theory has been summed up to Galois associations and Grothendieck's Galois theory.

Galois' theory begun in the investigation of symmetric capacities – the coefficients of a monic polynomial are (up to sign) the rudimentary symmetric polynomials in the roots. For example, (x – a)(x – b) = x2 – (a + b)x + abdominal muscle, where 1, a + b and stomach muscle are the rudimentary polynomials of degree 0, 1 and 2 out of two factors.

This was first formalized by the sixteenth century French mathematician François Viète, in Viète's formulas, for the instance of positive genuine roots. In the conclusion of the eighteenth century British mathematician Charles Hutton,[2] the outflow of coefficients of a polynomial as far as the roots (not just for positive roots) was first comprehended by the seventeenth century French mathematician Albert Girard; Hutton composes:

the primary individual who comprehended the general precept of the development of the coefficients of the forces from the aggregate of the roots and their items. He was the primary who found the tenets for summing the forces of the underlying foundations of any equation.

In this vein, the discriminant is a symmetric capacity in the roots that reflects properties of the roots – it is zero if and just if the polynomial has a various root, and for quadratic and cubic polynomials it is sure if and just if all roots are genuine and particular, and negative if and just if there is a couple of unmistakable complex conjugate roots. See Discriminant:Nature of the roots for points of interest.

The cubic was first incompletely tackled by the 15– sixteenth century Italian mathematician Scipione del Ferro, who did not anyway distribute his outcomes; this technique, however, just understood one kind of cubic equation. This arrangement was then rediscovered freely in 1535 by Niccolò Fontana Tartaglia, who imparted it to Gerolamo Cardano, requesting that he not distribute it. Cardano at that point stretched out this to various different cases, utilizing comparable contentions; see more subtle elements at Cardano's strategy. After the disclosure of Ferro's work, he felt that Tartaglia's technique was not any more mystery, and subsequently he distributed his answer in his 1545 Ars Magna.[3] His understudy Lodovico Ferrari settled the quartic polynomial; his answer was likewise incorporated into Ars Magna. In this book, be that as it may, Cardano does not give a "general formula" for the arrangement of a cubic equation, as he had neither one of the complexs numbers available to him, nor the arithmetical documentation to have the capacity to depict a general cubic equation. With the advantage of present day documentation and complex numbers, the formulae in this book do work in the general case, however Cardano did not know this. It was Rafael Bombelli who figured out how to see how to function with complex numbers with the end goal to explain all types of cubic equation.

A further advance was the 1770 paper Réflexions sur la résolution algébrique des équations by the French-Italian mathematician Joseph Louis Lagrange, in his strategy for Lagrange resolvents, where he broke down Cardano and Ferrarri's answer of cubics and quartics by considering them as far as changes of the roots, which yielded an assistant polynomial of lower degree, giving a brought together comprehension of the arrangements and laying the basis for gathering theory and Galois theory. Significantly, nonetheless, he didn't consider structure of changes. Lagrange's technique did not stretch out to quintic equations or higher, on the grounds that the resolvent had higher degree.

The quintic was nearly demonstrated to have no broad arrangements by radicals by Paolo Ruffini in 1799, whose key knowledge was to utilize change groups, not only a solitary stage. His answer contained a hole, which Cauchy considered minor, however this was not fixed until crafted by Norwegian mathematician Niels Henrik Abel, who distributed a proof in 1824, subsequently building up the Abel– Ruffini hypothesis.

While Ruffini and Abel built up that the general quintic couldn't be understood, some specific quintics can be tackled, for example, (x − 1)5 = 0, and the exact model by which a given quintic or higher polynomial could be resolved to be reasonable or not was given by Évariste Galois, who demonstrated that whether a polynomial was feasible or not was identical to regardless of whether the change gathering of its underlying foundations – in present day terms, its Galois gathering – had a specific structure – in current terms, regardless of whether it was a reasonable gathering. This gathering was constantly resolvable for polynomials of degree four or less, however not generally so for polynomials of degree five and more noteworthy, which clarifies why there is no broad arrangement in higher degree.

1.1 Galois' writings

In 1830 Galois (at 18 years old) submitted to the Paris Academy of Sciences a journal on his theory of feasibility by radicals; Galois' paper was at last rejected in 1831 as being excessively crude and for giving a condition as far as the underlying foundations of the equation rather than its coefficients. Galois then passed on in a duel in 1832, and his paper, "Mémoire surfles conditions de résolubilité des équations standard radicaux", stayed unpublished until the point when 1846 when it was distributed by Joseph Liouville joined by his very own portion explanations.[4] Prior to this production, Liouville declared Galois' outcome to the Academy in a discourse he gave on 4 July 1843.[5] According to Allan Clark, Galois' portrayal "drastically overrides crafted by Abel and Ruffini."[6]

1.2 Aftermath

Galois' theory was famously troublesome for his peers to see, particularly to the dimension where they could develop it. For instance, in his 1846 discourse, Liouville totally missed the gathering theoretic center of Galois' method.[7] Joseph Alfred Serret who gone to a portion of Liouville's discussions, incorporated Galois' theory in his 1866 (third release) of his reading material Cours d'algèbre supérieure. Serret's student, Camille Jordan, had a far and away superior understanding reflected in his 1870 book Traité des substitutions et des équations algébriques. Outside France, Galois' theory stayed more dark for a more extended period. In Britain, Cayley neglected to get a handle on its profundity and famous British variable based math course readings did not make reference to Galois' theory until well after the turn of the century. In Germany, Kronecker's works centered more around Abel's outcome. Dedekind composed minimal about Galois' theory, however addressed on it at Göttingen in 1858, demonstrating a decent understanding.[8] Eugen Netto's books of the 1880s, in light of Jordan's Traité, made Galois theory open to a more extensive German and American gathering of people as did Heinrich Martin Weber's 1895 variable based math textbook.[9]

1.3 INVERSE GALOIS PROBLEM

In Galois theory, the inverse Galois issue concerns regardless of whether each finite gathering shows up as the Galois gathering of some Galois expansion of the normal numbers Q. This issue, first presented in the mid nineteenth century,[1] is unsolved.

There are some stage groups for which conventional polynomials are known, which characterize every single mathematical augmentation of Qhaving a specific gathering as Galois gathering. These groups incorporate all of degree no more prominent than 5. There additionally are groups known not to have nonexclusive polynomials, for example, the cyclic gathering of request 8.

All the more by and large, given G a chance to be a given finite gathering, and given K a chance to be a field. At that point the inquiry is this: is there a Galois expansion field L/K with the end goal that the Galois gathering of the augmentation is isomorphic to G? One says that G is feasible over K if such a field L exists.

1.4 Incomplete outcomes

There is a lot of point by point data specifically cases. It is realized that each finite gathering is feasible over any capacity field in one variable over the perplexing numbers C, and all the more by and large over capacity fields in a single variable over any arithmetically shut field of trademark zero. Igor Shafarevich demonstrated that each finite resolvable gathering is feasible over Q.[2] It is likewise realized that each sporadic gathering, aside from conceivably the Mathieu aggregate M23, is feasible over Q.[3]

David Hilbert had demonstrated that this inquiry is identified with a discernment question for G:

On the off chance that K is any augmentation of Q, on which G goes about as an automorphism gathering and the invariant field KG is normal over Q, at that point G is feasible over Q.

Here sane implies that it is an absolutely supernatural augmentation of Q, produced by a logarithmically autonomous set. This foundation can for instance be utilized to demonstrate that all the symmetric groups are feasible.

Much point by point work has been done on the inquiry, which is in no sense settled as a rule. A portion of this depends on developing G geometrically as a Galois covering of the projective line: in arithmetical terms, beginning with an expansion of the field Q(t) of discerning capacities in an uncertain t. From that point forward, one applies Hilbert's immutability hypothesis to practice t, so as to safeguard the Galois gathering.

All change groups of degree 16 or less are known to be feasible over Q [4]; the gathering PSL(2,16):2 of degree 17 may not be [5].

Each of the 13 non-Abelian straightforward groups littler than PSL(2,25) (arrange 7800) are known to be feasible over Q. [6]

1.5 Del Pezzo surfaces over finite fields

Let X be a del Pezzo surface of degree 2 or greater over a finite field Fq. The image Γ of the Galois group Gal($\bar{F}_q$/Fq) in the group Aut(Pic($\bar{X}$)) is a cyclic subgroup preserving the anticanonical class and the intersection form. The conjugacy class of Γ in the subgroup of Aut(Pic($\bar{X}$)) preserving the anticanonical class and the intersection form is a natural invariant of X. We say that the conjugacy class of Γ in Aut(Pic($\bar{X}$)) is the \textit{type} of a del Pezzo surface. In this paper we study which types of del Pezzo surfaces of degree 2 or greater can be realized for given q. We collect known results about this problem and fill the gaps.

II. Related work

Our IGT picture. The graph next page outlines the objective zone and the points of reference of our venture. We allude to our glossary in x3 for a full de nition of our abbre-viated IGT proclamations (which specialists will effortlessly perceive). All ramifications appeared on the image are either established or simple.

There are two primary divisions in our image: vertical and level.

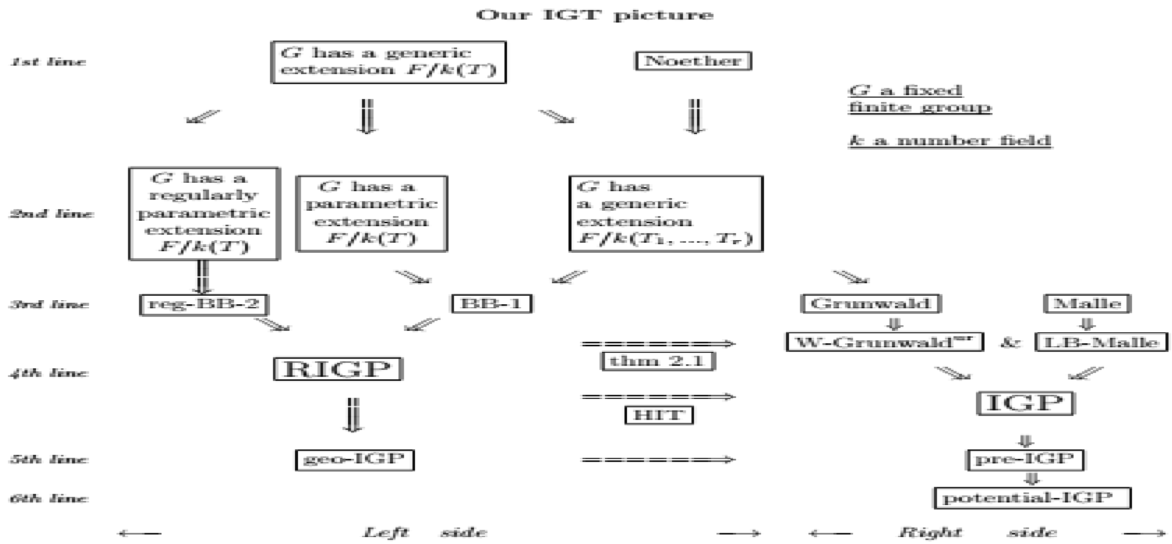
2.1 Vertical division. Left side comprises of geometric articulations while right side is con-cerned with math ones. Nearness of indeterminates is the acknowledgment indication of the previous. Specialization interfaces the opposite sides. We practice a k-ordinary Galois expansion F=k(T) or the corresponding k-cover f : X ! P^1_T in two ways[1]:

- for t_0 2 k, F_{t0} =k, also denoted by f_{t0} , is the classical specialized extension of F at t_0:

the residue eld extension at some prime ideal above t_0 in the extension F=k(T).

- if T_0 2 k(U) n k, f_{T0} : X_{T0} ! P^1_U is the pull-back of f along T_0 : P^1 ! P^1, which corre-sponds to specializing T to T_0(U). We use the pre x G for this G(eometric) specialization, which stays on the left side

2.2 Hilbert Irreducibility Theorem (HIT) is the crucial specialization apparatus making the con-nection among geometric and number juggling sides, eminently demonstrating R(egular)IGP) IGP.



Even division. The fourth line ramifications RIGP) IGP between the two Inverse Galois Problems (Regular and traditional) is the isolating line. Articulations above it are solid types of the two authentic explanations. They likely could be very solid. Discrediting them and subsequently defining limits to reality in IGT would be an rst objective. For the greater part of them, even extremely solid ones, it is a main problem. For every one of the, a greater objective is to research further (preferably order) which groups fulfill the announcements and which don't. Explanations beneath RIGP) IGP are feeble types of the issues, perhaps more conceivable and open. Demonstrating them this time is an objective of the undertaking.

III. SURFACES — PROOF OF THEOREM 1.1

As explained in the introduction, by [15, Theorem 1.3] it suffices to construct the remaining case C_{14} over every finite field. To do so we require the following lemma.

Lemma 2.1. Let k be an algebraically closed field and let Π_1, Π_2 and Π_3 be three planes in P^3_k which have exactly one common point Q. Let C_1, C_2 and C_3 be three cubic curves lying in Π_1, Π_2 and Π_3 respectively, such that $\Pi_i \cap C_j = \Pi_j \cap C_i$ and such that Q ∈/ C_i for any i, j. Then there is one-dimensional family of cubic surfaces over k containing C_1, C_2 and C_3.

Proof. We can assume the planes are given by $\Pi_i : x_i = 0$, so Q = (1 : 0 : 0 : 0).

The cubic curves are given by $C_i : P_i(x_0, \dots, xb_i, \dots, x_3) = 0$. Each polynomial P_i contains the monomial $x^3{}_0$, since Q ∈/ C_i, so we may assume that the coefficient of $x^3{}_0$ is 1 for each polynomial P_i. Put

$P(x_0, x_1, x_2, x_3) = P_1(x_0, x_2, x_3) + P_2(x_0, x_1, x_3) + P_3(x_0, x_1, x_2)$

$P_1(x_0, 0, x_3) - P_2(x_0, x_1, 0) - P_3(x_0, 0, x_2) + x^3{}_0$.

Then one has

$P(x_0, 0, x_2, x_3) = P_1(x_0, x_2, x_3), P(x_0, x_1, 0, x_3) = P_2(x_0, x_1, x_3)$,

$P(x_0, x_1, x_2, 0) = P_3(x_0, x_1, x_2)$, since $P_1(x_0, 0, 0) = P_2(x_0, 0, 0) = P_3(x_0, 0, 0) = x^3{}_0$ and

$P_1(x_0, 0, x_3) = P_2(x_0, 0, x_3)$, $P_1(x_0, x_2, 0) = P_3(x_0, 0, x_2)$, $P_2(x_0, x_1, 0) = P_3(x_0, x_1, 0)$

Therefore for any $(\mu : \lambda) \in P^1_k$ the equation $\mu x_1x_2x_3 = \lambda P(x_0, x_1, x_2, x_3)$ gives a cubic surface passing through the curves C_1, C_2 and C_3.

We apply this as follows. Let F_q be a finite field and F the Frobenius automorph- ism. Let L1, L2 and L3 be a triple of conjugate non-coplanar lines in P^3 passing F_Q3

through a common point $Q \in P^2(F_q)$. Consider a point $p_1 \in L_1(F_q9) \setminus L_1(F_q3)$, and its conjugates $p_{i+1} = F^i p_1$, $i \in \{2, . . . , 9\}$. One has $p_i, p_{i+3}, p_{i+6} \in L_i$ (here and in what follows, subscripts are taken modulo 9). Consider the line E_1 passing through

p_1 and p_2, and its conjugates $E_{i+1} = F^i E_1$, $i \in \{2, . . . , 9\}$. Denote this configuration of nine conjugate lines by E. We want to show that there exists a smooth cubic surface X over F_q containing E.

The configuration E is defined over F_q, in particular the linear system of cubics containing it is defined over F_q. For $\{i, j, k\} = \{1, 2, 3\}$, let Π_i be the plane spanned

by L_j and L_k. Then $E_{i+1}, E_{i+4}, E_{i+7} \subset \Pi_i$ and $\Pi_i \cap \Pi_j \cap E = \{p_k, p_{k+3}, p_{k+6}\}$. Therefore by Lemma 2.1 there is one-dimensional family X of cubic surfaces over F_q passing through E. We now show that there is a unique singular member.

Lemma 2.2. There is a unique singular cubic surface over F_q containing the con- figuration E, given by the union of the planes Π_1, Π_2 and Π_3.

Proof. Let X be a singular surface containing E, and S a singular point of X_{F_Q} . There is a line E in E that does not pass through S. Without loss of generality, we may assume that $E = E_1$. Then E_1 meets the other lines in E in at least 3 points, as it meets the lines E_2, E_9 and E_4 in distinct points. Thus any line in P^2 passing through one of these points and S meets both points in multiplicity at least 2, hence is contained in X. Thus the plane spanned by E and S contains at least 4 lines in X, therefore this plane lies in X. As the lines E_i are conjugate, the only possibility is that X is the union of planes of planes Π_1, Π_2 and Π_3.

So by Lemmas 2.1 and 2.2 there exists a smooth cubic surface X containing the configuration E, as X has $q + 1 \geq 3$ elements. Any such cubic surface contains nine conjugate lines, therefore X has type C_{14}, since it is the only conjugacy class of $W(E_6)$ consisting of elements of order divisible by 9 in [12, Tab. 1].

IV. DEL PEZZO SURFACES OF DEGREE 2 — PROOF OF THEOREM 1.2

By Theorem 1.1, over every finite field F_q there exists a cubic surface with class C_{14}. Such a surface has a rational point. Moreover, by a consideration of the Galois action on the lines, no rational point lies on a line over F_q. Thus the blow-up of a rational point is a del Pezzo surface of degree 2. This constructs the missing class 47 over F_q. One then performs a Geiser twist to get the class 56.

By [15, Theorem 1.2], this leaves open the existence of classes 28 and 35 over F_3. These surfaces are Geiser twists of each other, thus it suffices to construct the class 35. We claim that this is realised by the surface

$$X : (x^2 + xz - z^2)s^2 + (x^2 + y^2)st + (x^2 - xy - y^2 + xz - z^2)t^2 = 0 \subset P^1_{s,t} \times P^2_{x,y,z}.$$

We first prove that X is a del Pezzo surface of degree 2. By the adjunction formula the anticanonical bundle is O_X (0, 1), thus the anticanonical map is given by the projection onto the second factor X → P^2. This map is proper and a simple calculation shows that it is quasi-finite, hence it is finite. Thus the ant canonical bundle, being the pull-back of an ample line bundle by a finite morphism, is ample. Thus X is del Pezzo, and from the equation it is obvious that it has degree 2.

Next, the projection onto the first factor equips X with the structure of a conic bundle. The singular fibres lie over the closed points of P^1 given by s = 0, t = 0, $s^2 + t^2 = 0$ and $s^2 + st - t^2 = 0$, moreover one checks that these singular fibres are all irreducible, so that X → P^1 is relatively minimal. Therefore from [8, Thm. 2.6], one finds that the Frobenius element acts on the Picard group Pic X with eigenvalues 1, −1, −1, i, −i, i, −i. However, an inspection of [14, Appendix, Tab. 1] reveals that class 35 is the only one with this property; hence the class is 35, as claimed.

Remark 3.1. In fact any del Pezzo surface of degree 2 with a conic bundle structure arises as a surface of bidegree (2, 2) in $P^1 \times P^2$; see [5, Thms. 5.6, 5.7].

REFERENCES

1. Pierre Debes. On the Malle conjecture and the self-twisted cover. Israel J. Math., 218:101{131, 2017.
2. Pierre Debes. Groups with no parametric Galois realization. Annales Sci. E.N.S., 51/1:143{ 179, 2018.
3. B. Banwait, D. Loughran and F.Fit´e, Del Pezzo surfaces over finite fields and their Frobenius traces. *Math. Proc. Cambridge Philos. Soc.*, to appear.
4. R. W. Carter, Conjugacy classes in the Weyl group. *Comp. Mathematica*, 25, 1972, 1–59.
5. I.V. Dolgachev, *Classical algebraic geometry: A modern view*. Cambridge University Press, Cambridge, 2012.
6. A.-S. Elsenhans, J. Jahnel, Moduli spaces and the inverse Galois problem for cubic surfaces. *Trans. Amer. Math. Soc.* 367 (2015), no. 11, 7837–7861.
7. C. Frei, D. Loughran, E. Sofos, Rational points of bounded height on general conic bundle surfaces. *Proc. Lond. Math. Soc.* 117 (2018), 407–440.
8. Knecht, K. Reyes, Full degree two del Pezzo surfaces over finite fields. In: *Contemporary Developments in Finite Fields and Their Applications*, World Scientific Press, 145–159, 2016.
9. Y.I. Manin, *Cubic forms*. North-Holland Mathematical Library 4, 2nd ed. 1986.
10. S. Rybakov, Zeta functions of conic bundles and del Pezzo surfaces of degree 4 over finite fields. *Mosc. Math. J.* 5 (2005), no. 4, 919–926, 974.
11. S. Rybakov, A. Trepalin, Minimal cubic surfaces over finite fields. *Mat. Sb.* 208 (2017), no. 9, 148–170.
12. C. Salgado, D. Testa, A. V´arilly-Alvarado, On the Unirationality of del Pezzo surfaces of degree two. *J. Lond. Math. Soc.* 90 (2014), 121–139.
13. J.-P. Serre, *Lectures on* N_X (P). Chapman & Hall/CRC Res. Notes in Math., 11, 2012.
14. H.P.F. Swinnerton-Dyer, The zeta function of a cubic surface over a finite field. *Proc. Cam-bridge Philos. Soc.* 63 (1967) 55–71.
15. H.P.F. Swinnerton-Dyer, Cubic surfaces over finite fields. *Math. Proc. Cambridge Philos. Soc.* 149 (2010), 385–388.
16. Trepalin, Minimal del Pezzo surfaces of degree 2 over finite fields. *Bull. Korean Math. Soc.* 54 (2017), no. 5, 1779–1801.

17. Trepalin, Del Pezzo surfaces over finite fields. Preprint, http://arxiv.org/abs/1803.07421.
18. T. Urabe, Calculation of Manin's invariant for del Pezzo surfaces. *Math. Comp.* 65 (1996), no. 213, 247–258, S15–S23.