FUZZY LINEAR PROGRAMMING PROBLEMS OF FIRST TYPE

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Abstract :

In this paper, Fuzzy linear programming problems an optimization problem of maximizing the expected utility. When probabilities of the outcomes are not known, or may not even be relevant and out come for, each action are characterized only approximately, we say that decision are made, under uncertainty.

Introduction :

Linear programming problems isone of the most fundamental activities of human and inhuman beings. Linear programming problems is the study of how decisions are actually made and how they can be made better or more successfully. Linear programming problems plays an important role in economic and business management science, engineering and manufacturing, social and political science, biology and medicine, military strategy.

A problems is said to be made under conditions of certainly when the outcome for each action can be determined and ordered precisely. In this case the alternative that leads to the outcome yielding the highest utility is chosen. That is the decision making problem becomes an optimization problem, the problem of maximizing the utility function. A decision is made under conditions or risk, on the other hand, when the only available knowledge concerning the outcomes distributions, one of each action.

Classical Linear Programming Problem :

The classical linear programming problem is to find the minimum or maximum values of a linear function under constraints represented by inequalities or equations. The general form of classical linear programming problem is as follows:

Maximize (or Minimize) $z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$

subject to $a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n \le b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n \le b_2$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n \leq b_m$$

$$x_1, x_2, \ldots, x_n \geq 0.$$

The function z which is maximized (or minimized) is called an objective function. The numbers $c_i(i \in N_n)$ are called cost coefficients and the vector $c = \langle c_1, c_2, \ldots, c_n \rangle$ is called a cost vector. The matrix $A = [a_{ij}]$, where $i \in N_m$ and $j \in N_n$, is called a constraint matrix and the vector $b = \langle b_1, b_2, \ldots, b_m \rangle^T$ is called a right-hand-side vector. The formulation of the problem can be written in vector form as:

$$\text{Max} \quad z = cx$$

$$\text{subject to} \quad Ax \leq b \qquad \ldots \text{(i)}$$

$$x \geq 0,$$

where $x = \langle x_1, x_2, \ldots, x_n \rangle^T$ is a vectorof variables. The set of vectors x that satisfy all given constraints is called a feasible set.

Fuzzy Linear Programming Problem :

The most general type of fuzzy linear programming problem is formulated as follows:

$$\text{Max} \quad \sum_{j=1}^{n} C_j X_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} A_{ij} X_j \leq B_j \left(i \in N_m \right) \qquad \ldots \text{(ii)}$$

$$X_j \geq 0 \ (j \in N_n),$$

whereA_{ij}, B_i, C_j are fuzzy numbers, and X_j are variables whose states are fuzzy numbers ($i \in N_m$, $j \in N_n$); the operations of addition and multiplication are operations of fuzzy arithmetic and $\leq$ denote the ordering of fuzzy numbers. Let us discuss two special cases of fuzzy linear programming problems. '

Case I. Fuzzy linear programming problems in which only the right-hand-side numbers B_i, are fuzzy numbers:

$$\text{Max} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \leq B_j \left(i \in N_m\right) \qquad \text{… (iii)}$$

$x_j \geq 0$ ($j \in N_n$).

Case 2. Fuzzy linear programming problems in which the right-hand-side numbers B_i and the coefficients A_{ij} of the constraint matrix are fuzzy numbers:

$$\text{Max} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} A_{ij} x_j \leq B_i \left(i \in N_m\right) \qquad \text{… (iv)}$$

$X_j \geq 0$ ($j \in N_n$),

In general, fuzzy linear programming problems are first converted into equivalent crisp linear or nonlinear problems, which are then solved by standard methods. The final results of a fuzzy linear programming problem are thus real numbers, which represent a compromise in terms of the fuzzy numbers involved.

Fuzzy Linear Programming Problems of First Type :

In this case, fuzzy numbers B_i ($i \in N_m$) typically have the form

$$B_i(x) = \begin{cases} 1 & \text{when } x \leq b_i \\ \dfrac{b_i + p_i - x}{p_i} & \text{when } b_i < x < b_i + p_i \\ 0 & \text{when } b_i + p_i \leq x \end{cases}$$

where $x \in R$. For each vector $x = \langle x_1, x_2, \ldots, x_n \rangle$, we first calculate the degree, $D_i(x)$, to which x satisfies the ith constraint ($i \in N_m$) by the formula $D_i(x) = B_i\left(\sum_{j=1}^{n} a_{ij} x_j\right)$.

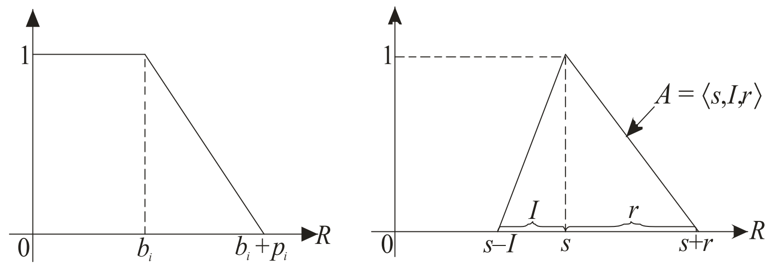
These degrees are fuzzy sets on R^n, and their intersection, $\bigcap_{i=1}^{m} D_i$, is a fuzzy feasible set.

Next, we determine the fuzzy set of optimal values. This is done by calculating the lower and upper bounds of the optimal values first. The lower bound of the optimal values, z_l is obtained by solving the standard linear programming problem:

Max $z = cx$

subject to $\sum_{j=1}^{n} a_{ij}x_j \le b_i \, (i \in N_m)$

$x_j \geq 0 \; (j \in N_n)$.



(a) Fuzzy numbers (b) Triangular fuzzy numbers

Fig.: Types of fuzzy numbers used in fuzzy linear programming problem: (a) fuzzy number in (iii) (b) triangular fuzzy numbers used in (iv)

The upper bound of the optimal values, z_u, is obtained by a similar linear programming problem in which each b_i is replaced with $b_i + p_i$:

Max $z = cx$

subject to $\sum_{j=1}^{n} a_{ij}x_j \le b_i + p_i \, (i \in N_m)$

$x_j \geq 0 \; (j \in N_n)$.

Then, the fuzzy set of optimal values, G, which is a fuzzy subset of R^n, is defined by

$$G(x) = \begin{cases} 1 & \text{when } z_u \le cx \\ \dfrac{cx - z_l}{z_u - z_l} & \text{when } z_l \le cx \le z_u \\ 0 & \text{when } cx \le z_l \end{cases}$$

Now, the problem (iii) becomes the following classical optimization problem:

Max λ

subject to $\lambda(z_u - z_l) - cx \le -z_l$

$$\lambda p_i + \sum_{i=1}^{n} a_{ij}x_j \le b_i + p_i \, (i \in N_m)$$

$\lambda, x_j \geq 0$ $(j \in N_n)$.

The above problem is actually a problem of finding $x \in R^n$ such that $\left[\left(\bigcap_{i=1}^{m} D_i\right) \cap G\right](x)$ reaches the maximum value; that is, a problem of finding a point which satisfies the constraints and goal with the maximum degree. The method used here is called a symmetric method (i.e., the constraints and the goal are treated symmetrically).

Ex. 1. Assume that a company makes two types of products. Product P_1 has aRs 0.40 per unit profit and product P_2 has a Rs 0.30 per unit profit. Each unit of product P_1 requires twice as many labour hours as each product P_2. The total available labour hours are at least 500 hours per day, and may possible be extended to 600 hours per day, due to special arrangements for overtime work. The supply of material is at least sufficient for 400 units of both P_1 and P_2 products per day, but may possibly be extended to 500 units per day according to previous experience. The problem is show that or find, how many units of product P_1, and P_2. should be made per day to maximize the total profit?

Sol. First, we calculate the lower and upper bound of the objective function by classical linear programming method:

Let x_1, x_2 denote the number of units of products P_1, P_2 made in one day, respectively.

Max $\quad z = 0.4x_1 + 0.3x_2$ (profit)

$x_1 + x_2 \leq 400$ (material)

subject to $\quad 2x_1 + x_2 \leq 500$ (labour hours)

$x_1, x_2 \geq 0$

Introducing the slack variables x_3 and x_4, we can write

Max $\quad z_I = 0.4x_1 + 0.3x_2 + 0.x_3 + 0.x_4$

subject to $\quad x_1 + x_2 + x_3 = 400$

$2x_1 + x_2 + x_4 = 500$

$x_1, x_2, x_3, x_4 \geq 0$

The solution to the problem is obtained by using simplex method.

Table : Simplex method

B	C_B	C_J→	0.4	0.3	0	0	Min. Ratio x_B/x_j
		x_B ↓	x_1	x_2	x_3	x_4	
x_3 x_4	0 0	400 500	1 2	1 1	1 0	0 1	400 250→
$z=C_B{\cdot}x_B=$ 0		$\Delta_j = c_j - C_B x_j$	0.4 ↑	0.3	0	0 ↓	
x_3 x_1	0 0.4	150 250	0 1	1/2 1/2	1 0	–1/2 1/2	75→ 125
z =100		Δ_j	0	0.1 ↑	0 ↓	–0.2	
x_2 x_1	0.3 0.4	300 100	0 1	1 0	2 –1	–1 1	
z =130		Δ_j	0	0	–0.2	–0.1	

Since all $\Delta_j \leq 0$, therefore the solution is optimal. Hence, the optimal solution is $x_1 = 100$, $x_2 = 300$, $z_l = 130$.

Now, we find the upper bound of the objective function :

Max $\quad z_u = 0.4x_1 + 0.3x_2$

subject to $\quad x_1 + x_2 \leq 500$

$2x_1 + x_2 \leq 600$

$x_1, x_2 \geq 0.$

Introducing the slack variables x_3 and x_4, we have :

Max $\quad z_u = 0.4x_1 + 0.3x_2 + 0.x_3 + 0.x_4$

subject to $\quad x_1 + x_2 + x_3 = 500$

$2x_1 + x_2 + x_4 = 600$

$x_1, x_2, x_3, x_4 \geq 0.$

The solution to the problems is obtained by using simplex method.

Table : Simplex method

B	C_B	c_j→	0.4	0.3	0	0	Min. Ratio
		x_B ↓	x_1	x_2	x_3	x_4	
x_3 x_4	0 0	500 600	1 2	1 1	1 0	0 1	500 300→

$z = 0$		Δ_j	0.4 ↑	0.3	0	0 ↓	
x_3	0	200	0	1/2	1	–1/2	100→
x_1	0.4	300	1	1/2	0	1/2	150
$z = 120$		Δ_j	0	0.1 ↑	0 ↓	–0.2	
x_2	0.3	400	0	1	2	–1	
x_1	0.4	100	1	0	–1	1	
$z_u = 160$		Δ_j	0	0	–0.2	–0.1	

Since all $\Delta_j \leq 0$, therefore the solution is optimal. Hence, the optimal solution is $x_1 = 100$, $x_2 = 400$, $z_u = 160$.

Fuzzy linear programming problem becomes :

$$\text{Max} \quad z = \lambda$$

$$\text{subject to} \quad (160 - 130)\lambda - (0.4x_1 + 0.3x_2) \leq -130$$

$$30\lambda - (0.4x_1 + 0.3x_2) \leq -130$$

$$-30\lambda + (0.4x_1 + 0.3x_2) \leq 130$$

$$100\lambda + x_1 + x_2 \leq 500$$

$$100\lambda + 2x_1 + x_2 \leq 600$$

$$\lambda, x_1, x_2 \geq 0.$$

Applying Big-M method, we have

$$\text{Max} \quad z = \lambda + 0.x_1 + 0.x_2 + 0.x_3 + 0.x_4 + 0.x_5 - M.A$$

$$\text{subject to} \quad 100\lambda + x_1 + x_2 + x_3 = 500$$

$$100\lambda + 2x_1 + x_2 + x_4 = 600$$

$$-30\lambda + 2/5x_1 + 3/10x_2 + x_5 + A = 130$$

$$\lambda, x_1, x_2, x_3, x_4, x_5\, A \geq 0.$$

The solution to the problems is obtained by using Big-M method.

Table : Big-M method

B	C_B	C_j→	1	0	0	0	0	0	$-M$	Min. Ratio
		x_B ↓	λ	x_1	x_2	x_3	x_4	x_5	A	

x_3	0	500	100	1	1	1	0	0	0	500
x_4	0	600	100	2	1	0	1	0	0	300→
A	$-M$	130	–30	2/5	3/10	0	0	–1	1	325
$z =$	$130M$	Δ_j	$1 - 30M$	$2M/5$ ↑	$3M/10$	0	0 ↓	$-M$	0	
x_3	0	200	50	0	1/2	1	–1/2	0	0	400
x_1	0	300	50	1	1/2	0	1/2	0	0	600
A	$-M$	10	–50	0	1/10	0	–1/5	–1	1	100→
$z =$	$-10M$	Δ_j	$1 - 50M$	0	$M/10$ ↑	0	$-M/5$	$-M$	0 ↓	
x_3	0	150	300	0	0	1	1/2	5		1/2→
x_1	0	250	300	1	0	0	3/2	5		5/6
x_2	0	100	–500	0	1	0	–2	–10		–
$z =$	0	Δ_j	1 ↑	0	0	0 ↓	0	0		
λ	1	1/2	1	0	0	1/300	1/600	1/60		1/2→
x_1	0	100	0	1	0	–1	1	0		5/6
x_2	0	350	0	0	1	5/3	–7/5	–5/3		
		Δ_j	0	0	0	–1/300	–1/600	–1/60		

Since all $\Delta_j \leq 0$, therefore the solution is optimal. Hence, the optimal solution is $\lambda = 0.5$, $x_1 = 100$, $x_2 = 350$.

The maximum profit is $z = 145$.

Ex.2 Solve the following fuzzy linear programming problem

Max $z = 0.5x_1 + 0.2x_2$

subject to $x_1 + x_2 \leq B_1$

$2x_1 + x_2 \leq B_2$

$x_1, x_2 \geq 0.$

where,

$$B_1(x) = \begin{cases} 1 & \text{for } x \leq 300 \\ \dfrac{400 - x}{100} & \text{for } 300 < x \leq 400 \\ 0 & \text{for } x \leq 400 \end{cases}$$

and

$$B_2(x) = \begin{cases} 1 & \text{for } x \leq 400 \\ \dfrac{500 - x}{100} & \text{for } 400 < x \leq 500 \\ 0 & \text{for } x \leq 500 \end{cases}$$

Sol. We have to find the lower and upper bound of the objective function. The classical linear programming problem corresponding to the fuzzy linear programming problem is given by

Max $z_l = 0.5x_1 + 0.2x_2$

subject to $x_1 + x_2 \le 300$

$2x_1 + x_2 \le 400$

$x_1, x_2 \ge 0$

Introducing the slack variables x_3 and x_4, we can write

Max $z_l = 0.5x_1 + 0.2x_2 + 0.x_3 + 0.x_4$

subject to $x_1 + x_2 + x_3 = 300$

$2x_1 + x_2 + x_4 = 400$

$x_1, x_2, x_3, x_4 \ge 0.$

The solution to the problem is obtained by using simplex method.

Table : Simplex method

B	C_B	$c_j \rightarrow$	0.5	0.2	0	0	Min. Ratio
		$x_B \downarrow$	x_1	x_2	x_3	x_4	
x_3	0	300	1	1	1	0	300
x_4	0	400	2	1	0	1	200→
$z = 0$		Δ_j	0.5 ↑	0.2	0	0 ↓	
x_3	0	100	0	1/2	1	1/2	
x_1	0.5	200	1	1/2	0	1/2	
z_l=100		Δ_j	0	–0.5	0	–0.25	

Since all $\Delta_j \le 0$, therefore the solution is optimal. Hence, the optimal solution is $x_1 = 200$, $x_2 = 0$ and $z_l = 100$.

Now, we find the upper bound of the objective function :

Max $z_u = 0.5x_1 + 0.2x_2 + 0.x_3 + 0.x_4$

subject to $x_1 + x_2 + x_3 = 400$

$2x_1 + x_2 + x_4 = 500$

$x_1, x_2, x_3, x_4 \ge 0$

Table : Simplex method

B	C_B	$c_j \rightarrow$	0.5	0.2	0	0	Min. Ratio
		$x_B \downarrow$	x_1	x_2	x_3	x_4	
x_3	0	400	1	1	1	0	400
x_4	0	500	2	1	0	1	250→
$z = 0$		Δ_j	0.5 ↑	0.2	0	0 ↓	
x_3	0	150	0	1/2	1	–1/2	
x_1	0.5	250	1	1/2	0	1/2	
z_u=125		Δ_j	0	–0.05	0	–0.25	

Since all $\Delta_j \leq 0$, therefore the solution is optimal. Hence, the optimal solution is $x_1 = 250$, $x_2 = 0$, and $z_u = 125$.

Thus, the fuzzy linear programming problem becomes :

Max $z = \lambda$

subject to $(125 - 100)\lambda - (0.5x_1 + 0.2x_2) \leq -100$

$100\lambda + x_1 + x_2 \leq 400$

$100\lambda + 2x_1 + x_2 \leq 500$

$\lambda, x_1, x_2 \geq 0.$

The solution of this fuzzy linear programming problem is obtained by using Big-M Method:

Max $z = \lambda + 0.x_1 + 0.x_2 + 0.x_3 + 0.x_4 + 0.x_5 - M.A$

subject to $100\lambda + x_1 + x_2 + x_3 = 400$

$100\lambda + 2x_1 + x_2 + x_4 = 500$

$-25\lambda + 1/2x_1 + 1/5x_2 - x_5 + A = 100$

Table : Big-M method

B	C_B	$c_j \rightarrow$	1	0	0	0	0	0	$-M$	Min. Ratio
		$x_B \downarrow$	λ	x_1	x_2	x_3	x_4	x_5	A	
x_3	0	400	100	1	1	1	0	0	0	400
x_4	0	500	100	2	1	0	1	0	0	250
A	$-M$	100	–25	1/2	1/5	0	0	–1	1	200→
		Δ_j	$1 - 25M$	$M/2$ ↑	$M/5$	0	0	$-M$	0 ↓	

x_3	0	200	150	0	3/5	1	0	2		4/3
x_4	0	100	200	0	1/5	0	1	4		1/2→
x_1	0	200	–50	1	2/5	0	0	–2		–
		Δ_j	1 ↑	0	0	0	0 ↓	0		
x_3	0		0	0	9/20	1	–3/4	–1		
λ	0	1/2	1	0	1/1000	0	1/200	1/50		
x_1	0		0	1	9/20	0	1/4	–1		
		Δ_j	0	0	–1/1000	0	–1/200	–1/50		

Since all $\Delta_j \leq 0$, so obtained solution is optimal. Hence, the solution of fuzzy linear programming problem is given by

$$x_1 = 225,\ x_2 = 0,\ \lambda = 0.5.$$

Finally, maximum profit is given by

$$\text{Max} \quad z = 0.5x_1 + 0.2x_2 = 0.5 \times 225 = 112.5.$$

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