Heuristic Investigation of Mathematical Contribution for the Analysis and Design of Control System

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Abstract: The main focus of this paper is to investigate the mathematical contribution for the analysis and design of control system and thus make the class room teaching more interesting. History of engineering science is replete with the example of many complex topics which without illustrating its evolutionary details make the subject boring and monotonous. So here in this paper, a few examples are taken to substantiate the essence of the topic. For example the evolutionary details of optimal control, Laplace transform and Nyquist stability criterion, interval polynomial etc. has been discussed. This paper is culmination of practical experience as to how the evolutionary details of complex subject makes the lecture pedagogically stimulating. Eventually the arousal of natural interest in every topic is the basic purpose of any teacher and in that direction a novel idea has been conceptualized through this paper.

Keywords: Brachistochrone problem, Calculus of variation, Lagrange- Euler equation, Laplace transform, Nyquist, Cauchy's Mapping Theorem, Interval polynomial

1. Introduction

From time immemorial our ancestors faced many predicaments and hardships and endeavoured diligently in an unswerving way which eventually culminated in the form of modern science and technology. We have reached to this amazing level of science only exploiting a negligible fraction of this huge population. If negligible fraction of population could bring about this form of scientific spectacle then what will happen if substantive fraction of human population may synergistically be energized to pay undivided attention towards science and technology. It has been observed that in order to understand the complex aspect of science and technology, we require paradigm change in the pedagogy. It is worth noting that if we really want the education of science and technology accessible to larger portions of masses then we need to kindle the interest towards science. In order to kindle the interest of students towards science we have to make him/her understand even the complex aspect of science in a simpler and stimulating way [1-3].

One great physicist Henry Poincare once said- "We study nature because nature is interesting, if it were not interesting why would have we studied it". But ironically many students are not finding study of science interesting because of lack of the historical background of the complex topic which adds a new cap to feather and makes the class teaching highly motivating and stimulating. This is aim of good pedagogy too. It is conviction of many pragmatic souls that even the complex topic may be comprehended by taking recourse to simple examples and providing evolutionary background of the sophisticated topic. It is commonly said that Newton like scientist are not made rather they are born. So likewise each person is not ordained with this sort of ingenious and dexterous brain but even so their dormant talent may be stirred up by bringing about magical change in the pedagogy so that teachers may be accordingly trained towards making science funny, interesting and inquisitive. Here, by taking several examples it has been tried to portray that how the fear towards study of even the harder subject may be allayed completely by pedagogical improvement [4-6].

The design of modern control approach which takes into account time and energy minimization requires optimal controller. The evolution of optimal control approach has been engendered from calculus of variation [7-18]. If it was not for the spontaneous emergence of Brachistochrone problem in the mind of Johann Bernoulli, evolution of calculus of variation and hence the optimal control would not have been possible and all aforementioned potential realm of research would not have taken place [19, 23]. Thus the Brachistochrone problem heralded the evolution of optimal control law. The solution of the problem was found to be cycloid and hence known as the brachistochrone curve also by virtue of being the fastest curve [19-24].

Borschbach et al has presented the role of an evolutionary solution for the Brachistochrone problem [17]. Desiax et al has presented the evolutionary contribution of Brachistochrone problem for variational calculus [18]. Many authors have presented the work related to Brachistochrone problem, variational calculus and optimal control [25-32]. Variational calculus was later on found to be a powerful tool in mathematics and engineering and find potential applications in almost all realm of modern control engineering. The necessity of the subject can by no means be underrated notwithstanding the widespread application of computer in applied mathematics and engineering field. This was the potential reason as to why brachistochrone problem may be regarded as the mother of variational calculus which enchanted the entire scientific and mathematical world after its evolution in the late 17^{th} century [27-30].

In this paper, analogical interpretation and evolution of calculus of variation from classical differential calculus has been elegantly presented. Solution of a simple variational problem i.e. the functional of a single function requires from the fundamental principle of calculus of variation that the variation should be zero and that eventually culminates to Euler-Lagrange equation. In order to arrive at the Euler-Lagrange equation from fundamental principle of calculus of variation, everywhere in the literature, integrand has been expanded about the operating point in Taylor's series and subsequently the linear terms are retained and higher order terms are rejected [21, 22]. Doing this is very lengthy so taking analogy from the classical differential calculus a novel expression for variation has been elegantly arrived at which when equated to zero extracts the Euler-Lagrange equation in a few steps. As Euler-Lagrange equation plays crucial role in the application of calculus of variational approach to addressing the optimal control problem, this analogically derived expression for variation will expedite the work to a great extent [31-36].

2. Birth of optimal control from Brachistochrone problem

Calculus of variations paved the way for solving numerous kind of optimization problems encompassing prodigious fields. The very first person to conceptualize a problem which could easily have been solved by calculus of variation was Queen Dido of Carthage. Queen Dido was promised all the land she could at the most enclose with a bull's hides. It is said that she cleverly cut the bull hide into many piece and tied the end of all lengths together. After doing this, her next problem was to get the closed curve having fixed perimeter which must enclose maximum area [12]. The obvious answer is circle which satisfied the condition of the problem. Her problem could be easily solved by calculus of variations.

Another problem of historical importance is the Brachistochrone problem conceptualized by J. Bernoulli in 1696. The solution of the problem, a cycloid lying in the vertical plane, is credited to many mathematician such as Johann Bernoulli and Newton [22]. In Dido's and Brachistochrone problem, curves are desired that make some criterion to take minimum values. There is a great analogical link of this problem with the optimal control problem, where also the basic purpose is to seek a control function worthy to minimize a performance measure.

Modeling of Brachistochrone problem and consequently the solution is found by Euler-Lagrange equation. Hence, for checking the veracity of the solution curve, different other curves along with cycloid is taken, and time taken between two points, along all the curves are computed and compared. After comparison, it is found that the cycloid path takes least time.

2.1 Fundamental Concepts

There lies a great deal of analogy between differential calculus and variational calculus. In variational calculus or any kind of optimal control problems, the objective is to determine a function that optimizes a functional or performance measure. The corresponding analogous problem in differential calculus is to find a point that gives the minimum value of a function.

2.1.1 Definition of Function and Functional

A *function* f is a mapping from each element q of set D, to a unique element of set R. D is known as the *domain* of f and R is known as the *range*. Similarly, a *functional* J is a mapping from each function $\boldsymbol{x}$ of a certain class $\boldsymbol{\Omega}$ to a unique real number. $\boldsymbol{\Omega}$ is known as *domain* of the functional and the set of real number mapped with the functions in Ω is known as *range* of functional. Intuitively it can be said that a functional is a "*function of a function*"

2.1.2 Increment of a Function and Functional

In order to consider extreme value of a function, an increment is defined as

$$\Delta f(q, \Delta q) \triangleq f(q + \Delta q) - f(q) \quad (1)$$

Analogically, the increment of a functional J may be defined as

$$\Delta J(x, \Delta x) \triangleq J(x + \delta x) - J(x) \quad (2)$$

2.2 Analogy between Differential and Variation

The variation analogically plays the same role in finding extremal of a *functional* as the differential plays in determining the maxima and minima of functions.

2.2.1 Differential and Analogical definition of Variation

Fig. 1 gives a geometrical implication of the increment Δf, the differential df, and the derivative f'.

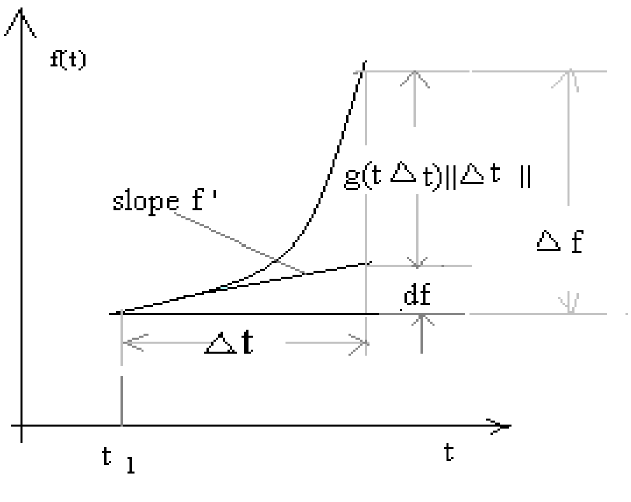

Fig. 1 Geometrical interpretation of Δf, df, f'

The *increment of a function* of n variable can be written as

$$\Delta f(q, \Delta q) = df(q, \Delta q) + g(q, \Delta q)\|\Delta q\| \quad (3)$$

and for one variable same can be written as

$$\Delta f(t,\Delta t) = df(t,\Delta t) + g(t,\Delta t)\|\Delta t\| \quad (4)$$

where df is linear function of Δt, if

$$\lim_{\|\Delta t\|\to 0}\{g(t,\Delta t)\} = 0 \quad (5)$$

Differential of f at point t and the differential can be expressed as

$$df(t,\Delta t) = f'(t)\Delta t \quad (6)$$

Figure 2.1 gives a geometrical implication of the increment Δf, differential df, and the derivative f'. Hence,

$$df(t,\Delta t) = \lim_{\|\Delta t\|\to 0}\Delta f(t,\Delta t) \quad (7)$$

In particular, if f is a function of n variables, the differential df is expressed as

$$df = \frac{\partial f}{\partial q_1}\Delta q_1 + \frac{\partial f}{\partial q_2}\Delta q_2 + \ldots\ldots\ldots + \frac{\partial f}{\partial q_n}\Delta q_n \quad (8)$$

It is convenient to evolve a formal analogical procedure for evolving the variation of a functional as given below.
The increment of a functional of n variable can be analogically written from eq. (3) as

$$\Delta J(x,\Delta x) = \delta J(x,\Delta x) + g(x,\Delta x)\|\Delta x\| \quad (9)$$

Where δJ is linear in δx. Now if

$$\lim_{\|\Delta x\|\to 0}\{g(x,\Delta x)\} = 0 \quad (10)$$

In equation (9), δJ is the variation of functional J. Analogical procedure for evolving the variation is developed as follows.

$$\delta J(x,\Delta x) = \lim_{\|\Delta x\|\to 0}\Delta J(x,\Delta x) \quad (11)$$

$$\delta J = \int_{t_0}^{t_f}\left[\frac{\partial J}{\partial x_1}\Delta x_1 + \frac{\partial J}{\partial x_2}\Delta x_2 + \ldots\ldots\ldots + \frac{\partial J}{\partial x_n}\Delta x_n\right]dt \quad (12)$$

2.3 Basic Theorem of Calculus of Variation

The basic theorem used in getting minima or maxima of functions stipulates that the differential be zero at an extreme point. The analogous theorem for getting extreme value of functional is that the variation must be zero on an extremal curve.
If x^* is an extremal curve, the variation of functional J must vanish on the extremal curve x^*

$$\delta J(x^*,\delta x) = 0 \quad (13)$$

This is nothing but the fundamental theorem of calculus of variation.

2.4 Application of Fundamental Theorem of Calculus of Variation to the Functional of a Single Function: Euler's Equation

Fundamental theorem of calculus of variation is used to find extremal curve for the functional of a single function as follows.

2.4.1 The Simplest Variational Problem

If x is a scalar function with continuous first derivatives. It is desired to find the extremal function x^* which minimizes the functional

$$J(x) = \int_{t_0}^{t_f} g(x(t),\dot{x}(t),t)dt \quad (14)$$

End points are fixed. In order to find the solution, the basic theorem of calculus of variation is applied. Revisiting eq. (13), and applying eq. (12) to the simplest problem (14) we get

$$\delta J = \int_{t_0}^{t_f}\left[\frac{\partial g}{\partial x}\delta x + \frac{\partial g}{\partial \dot{x}}\delta\dot{x}\right]dt \quad (15)$$

This after applying integral by parts to the second term, further implies

$$\delta J = \int_{t_0}^{t_f}\left[\frac{\partial g}{\partial x}\delta x\right]dt + \int_{t_0}^{t_f}\left[\frac{\partial g}{\partial \dot{x}}\delta\dot{x}\right]dt$$

$$= \int_{t_0}^{t_f}\left[\frac{\partial g}{\partial x}\delta x\right]dt + \int_{t_0}^{t_f}\left[\frac{\partial g}{\partial \dot{x}}\delta\dot{x}\right]dt$$

$$= \int_{t_0}^{t_f}\left[\frac{\partial g}{\partial x}\delta x\right]dt + \frac{\partial g}{\partial \dot{x}}|\delta x|_{t_0}^{t_f} - \int_{t_0}^{t_f}\left\{\frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right)\right\}\delta x(t)dt$$

$$= \frac{\partial g}{\partial \dot{x}}|\delta x|_{t_0}^{t_f} + \int_{t_0}^{t_f}\left[\frac{\partial g}{\partial x} - \frac{d}{dt}\left(\frac{\partial g}{\partial \dot{x}}\right)\right]\delta x(t)dt \quad (16)$$

For an extremal curve, the fundamental theorem on eq. (16), gives

$$\frac{\partial g}{\partial x}(x^*(t),\dot{x}^*(t),t) - \frac{d}{dt}\left\{\frac{\partial g}{\partial \dot{x}}(x^*(t),\dot{x}^*(t),t)\right\} = 0 \quad (17)$$

for all t belong to $[t_0, t_f]$. Equation (17) is called the Euler-Lagrange Equation. Here it is seen that the analogically derived expression (12) when applied to simple variational problem (14) generated the Euler's equation in a few step unlike the existing method of literature where, integrand is expanded about the operating point in Taylor's series and subsequently the linear terms are retained and higher order terms are rejected and thus becomes interminably lengthy. As Euler-Lagrange equation plays crucial role in the application of calculus of variational approach for addressing the optimal control problems, this analogically derived expression for variation will surely expedite the work to a great extent.
In the following section some important problems are solved by Euler-Lagrange equation.

3. FEW COMMON EXAMPLES TO TEST THE EFFICACY OF VARIATIONAL APPROACH

Here some interesting examples have been taken to highlight the indispensability of variational approach for solving these problems which otherwise could not have been solved by classical approach. Though the credit for the discovery of variational calculus goes to Brachistochrone problem, yet after its evolution it began to be fascinatingly applied to solve many problems though intriguing yet very practical and indispensable from the viewpoint of modern control design requirement.

Ex. 1: Smallest distance between two points in a plane
Arc length ds in a plane may be given as

$$ds = \sqrt{dx^2 + dy^2} \quad (15)$$

And the length of any curve going between points 1 & 2 is

$$J(y) = \int_1^2 ds \; = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \qquad (16)$$

Now using Euler equation for minimizing the functional J(y),

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \qquad (17)$$

where,

$$f = \sqrt{1 + \dot{y}^2} \qquad (18)$$

Now here we get $\partial f / \partial y = 0$, and

$$\frac{\partial f}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{1+\dot{y}^2}} \qquad (19)$$

and

$$\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = \frac{\ddot{y}}{\sqrt{1+\dot{y}^2}} \qquad (20)$$

Putting all these values from eq. (18) through eq. (20) into eq. (17) we get,

$$\ddot{y} = 0 \text{ or } \dot{y} = c_1$$

This implies that

$$y = c_1 x + c_2 \qquad (21)$$

This is an equation that represents a straight line.

Ex. 2: Minimum surface area of revolution

In order to form some surface of revolution, two fixed end points $A(x_1, y_1)$ and $B(x_2, y_2)$ are taken and revolved around the y axis. The problem is to get that curve which when revolved around y axis, generates a minimum surface area. The corresponding fig. 2 pertains to this problem.

The area of a small strip of the surface is

$$dA = 2\pi x \sqrt{1 + \dot{y}^2} dx \qquad (22)$$

and the total area is

$$A = 2\pi \int_A^B x\sqrt{1 + \dot{y}^2}\, dx \qquad (23)$$

applying Euler's equation again where

$$f = x\sqrt{1 + \dot{y}^2} \qquad (24)$$

We get, $\frac{\partial f}{\partial y} = 0$, and

$$\frac{\partial f}{\partial \dot{y}} = \frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} \qquad (25)$$

Hence from the Euler equation, we get

$$\frac{d}{dx}\left(\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}}\right) = 0 \qquad (26)$$

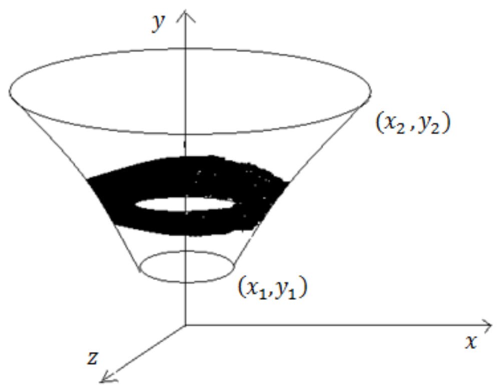

Fig. 2 Minimum surface of revolution

This after solution becomes

$$x = a\cosh\frac{y-b}{a} \qquad (27)$$

Which is the equation of a catenary. It is worth noting that these kind of fascinating mathematical problems would not have been solved without the evolution of variational calculus. In the next section Brachistochrone problem and its solution by variational approach is presented.

4. BRACHISTOCHRONE PROBLEM AND ITS SOLUTION BY VARIATIONAL APPROACH

4.1 Modeling of Brachistochrone Problem

Under the gravity, the bead moves between fixed end points 1 & 2. The problem here is to get the shape of the wire that makes bead to move from 1 to 2 in fig. 3 in minimum time.

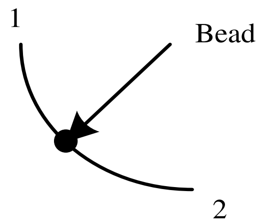

Fig. 3. Brachistochrone curve

The problem is then to minimize of the integral

$$t_{12} = \int_1^2 dt = \int_1^2 ds/v \qquad (28)$$

Energy of the particle can be written as $1/2mv^2 = mgy$ which implies that $v = \sqrt{2gy}$ hence from eq. (28) we obtain the functional as

$$t_{12} = \int_1^2 \sqrt{\frac{1+y'^2}{2gy}}\, dx \qquad (29)$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y} \qquad (30)$$

Where f is expressed as

$$f(y, y') = \sqrt{\frac{1+y'^2}{2gy}} \qquad (31)$$

This is required mathematical modeling for the solution of brachistochrone problem.

4.2 Solution of Brachistochrone Problem

Fom eq. (31) we get

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{y}\sqrt{1+y'^2}} \qquad (32)$$

hence

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{y''}{\sqrt{y}\sqrt{1+y'^2}} - \frac{y'^2}{2y^{3/2}\sqrt{1+y'^2}} - \frac{y'^2\, y''}{\sqrt{y}(1+y'^2)^{3/2}} \qquad (33)$$

and

$$\frac{\partial f}{\partial y} = -\frac{(1+y'^2)^{1/2}}{2y^{3/2}} \quad (34)$$

Which implies the solution as

$$x = -C + \frac{K}{2}(\alpha - \sin\alpha),\ y = \frac{K}{2}(1-\cos\alpha) \quad (35)$$

Equation (35) represents the equation of cycloid.
The motivation and inspiration for the design of modern controller such as optimal controller came from brachistochrone problem whose mathematical modeling involves functional.

5. Evolution of other Mathematical Topics of Interest

5.1 Evolution of Laplace and Z-Transform

As it is well known fact that Laplace and Z transform plays indispensable role for solving integro-differential and summation-difference equation. Evolution of Laplace and Z-Transform are evolved from Fourier series representation. But ironically for many students, in many book these two precious topics are just defined which doesn't give any idea about their evolution history which makes the students monotonous to these topics. In fact Fourier series representation in continuous time domain is extended to continuous time Fourier transform (CTFT) and then CTFT is further extended to Laplace Transform. Similarly, Fourier series representation in discrete time domain is extended to discrete time Fourier transform (DTFT) and then CTFT is further extended to Z Transform.

As it is well known fact that when we define any physical quantity, it means that it doesn't have any proof. For example angle is defined as the ratio of arc and radius i.e. $\theta = L/R$ which means that it is beyond proof. Similarly, in three dimensional geometry solid angle is defined as the ratio of small area on spherical surface and square of radius i.e. $dw = L/R^2$ also means that it has not any proof. Hence if Laplace and Z transform have proof of derivation then it is not justified to define it for engineering students because this way we are imposing something on him and make them losing their interest in such otherwise important and interesting topic.

5.2 Evolution of Nyquist stability criterion

Here attention is laid on another usefulness of complex variable for evolution of stability criterion in control system. In fact cauchy's mapping theorem also known as principle of argument is entirely a pure mathematics but harnessed elegantly by Nyquist to propound the stability theorem popularly known as Nyquist stability criterion. Principle of argument or Cauchy's Mapping theorem deals with the symbiotic relation between the mapping of closed loop s-plane contour and the corresponding $F(s)$ plane contour [34].

According to this theorem $N = Z - P$, where N is the number of encirclement of origin of $F(s)$ plane contour, Z is number of zeros and P is number of poles of $F(s)$ encircled by s-plane contour. This could be proved by the application of Cauchy's theorem and residue theorem. Cauchy's theorem is mathematically expressed as

$$\oint F(s)ds = 0$$

Where, $F(s)$ is analytic within and on the closed contour. If $F(s)$ is written as

$$F(s) = \frac{(s+z_1)^{k_1}(s+z_2)^{k_2}\ .\ .\ .}{(s+p_1)^{m_1}(s+p_2)^{m_2}\ .\ .\ .}X(s)$$

Where $X(s)$ is analytic in the closed contour in the s-plane all zeros and poles are located inside the contour. The ratio $F'(s)/F(s)$ could be written as

$$\frac{F'(s)}{F(s)} = \left(\frac{k_1}{s+z_1} + \frac{k_2}{s+z_2} + .\ .\ .\right) - \left(\frac{m_1}{s+p_1} + \frac{m_2}{s+p_2} + .\ .\ .\right) + \frac{X'(s)}{X(s)}$$

Residue theorem is mathematically expressed as

$$\oint \frac{F'(s)}{F(s)} = -2\pi j\left(\sum residues\right)$$

We have

$$\oint \frac{F'(s)}{F(s)} = -2\pi j[(k_1 + k_2 + .\ .\ .) - (m_1 + m_2 + .\ .\ .)]$$
$$= -2\pi j(Z - P)$$

F(s) can be written as

$$F(s) = |F|e^{j\theta}$$

and

$$lnF(s) = in|F| + j\theta$$

$F'(s)/F(s)$ could be written as

$$\frac{F'(s)}{F(s)} = \frac{d}{ds}\ln|F(s)|$$

We obtain

$$\frac{F'(s)}{F(s)} = \frac{d}{ds}\ln|F| + j\frac{d\theta}{ds}$$

or

$$\oint \frac{F'(s)}{F(s)}ds = \oint d\ln|F| + j\oint d\theta$$
$$= j\int d\theta = 2\pi j(P - Z)$$

Thus we get

$$\frac{\theta_2 - \theta_1}{2\pi} = P - Z$$

Noting that N is the clockwise encirclement of the origin of the F(s) plane, we obtain

$$\frac{\theta_2 - \theta_1}{2\pi} = -N$$

Whence we get

$$N = Z - P$$

Thus we see that how the mathematical concept of complex variable is harnessed later on by Nyquist to propound its own stability criterion. In the next section, the concept of interval polynomial and Kharitonov theorem is discussed.

5.3 Kharitonov's Theorem [36]

Implication of Kharitonov's Theorem

When the coefficients of characteristic equation are constant it has only one polynomial and stability may be easily found by Routh stability criterion. But when on the other hand, the coefficients of characteristics equation are not constants and assumes values within some interval, it leads to the formulation of interval polynomial. Interval polynomial constitutes infinite no of polynomial. Hence in order to study the robust stability of interval polynomial, Routh stability criterion has to be applied infinite times which become an onerous task. In order to deal with interval polynomial, Kharitonov theorem plays pivotal roles.
If a family of polynomials is given by

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n \quad (36)$$

where $\alpha_i \le a_i \le \beta_i$, $1 \le i \le n$, Kharitonov theorem says that the family of polynomials (36) will be stable if and only if the following four polynomials are stable:

$$s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \beta_3 s^{n-3} + \beta_4 s^{n-4} + \cdots \quad (37)$$

$$s^n + \alpha_1 s^{n-1} + \beta_2 s^{n-2} + \beta_3 s^{n-3} + \alpha_4 s^{n-4} + \cdots \quad (38)$$

$$s^n + \beta_1 s^{n-1} + \beta_2 s^{n-2} + \alpha_3 s^{n-3} + \alpha_4 s^{n-4} + \cdots \quad (39)$$

$$s^n + \beta_1 s^{n-1} + \alpha_2 s^{n-2} + \alpha_3 s^{n-3} + \beta_4 s^{n-4} + \cdots \quad (40)$$

6. Conclusions

As a conclusion, it can be inferred that the main focus of this paper is how to make class room teaching motivating and interesting by heuristic investigation of mathematical contribution for control system. History of engineering science is replete with the example of many complex topics which without illustrating its evolutionary details make the subject boring and monotonous. So here in this paper, a few examples are taken to substantiate the essence of the topic. For example the evolutionary details of optimal control, Laplace transform and Nyquist stability criterion etc. has been discussed. This paper is culmination of practical experience as to how the evolutionary details of complex subject makes the lecture pedagogically stimulating. Eventually the arousal of natural interest in every topic is the basic purpose of any teacher and in that direction a novel idea has been conceptualized through this paper.

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