QUATERNION ANALYTICITY OF TIME HARMONIC FIELD EQUATIONS FOR DYON IN HOMOGENEOUS MEDIA

Jivan Singh Garia[1], Chandra Mohan Singh Negi[2], Ila Joshi[3]

[1] Assistant Professor, Department of Physics, L. S. M. Govt. P.G. College, Pithoragarh, Uttarakhand

[2]Assistant Professor, Department of Physics, N.P S.I.T, Pithoragarh, Uttarakhand

[3] Research Scholar, Department of Physics, L. S. M. Govt. P.G. College, Pithoragarh, Uttarakhand

E-Mail: jgaria14@gmail.com, dr.cmsinghnegi@yahoo.in

ABSTRACT

Dirac put forward the idea of magnetic monopole while the fresh interest on the subject of monopoles was enhanced by the work of t'Hooft and Polyakov and its extension to the case of dyons by Julia and Zee. Consequently, these particles became intrinsic part of all current grand unified theories with their enormous potential importance in connection with various physical problems. Keeping in view the results of Witten, monopoles are necessarily dyons, a self-consistent and covariant quantum field theory for dyons each carrying the generalized charges as its real and imaginary parts has been constructed.

Starting with the Maxwell equations in presence of electric and magnetic sources in a homogeneous medium, we have derived the various quantum equations of dyons in consistent and manifest covariant way. It has been shown that the presented theory of dyons remain invariant under the Lorentz and duality transformation in homogeneous medium. In this paper, we have developed the quaternion analysis of time-dependent Maxwell-Dirac equations of dyons and this theory is extended consistently to time-dependent harmonic Maxwell equation of Dyons.

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Jivan Singh Garia[1] and Chandra Mohan Singh Negi[2]

[1] **Assistant Professor,** Department of Physics**,** L. S. M. Govt. P.G. College, Pithoragarh.(U. K).
[2]**Assistant Professor,** Department of Physics**,** N.P S.I.T, Pithoragarh, U..K.

E-Mail: jgaria14@gmail.com

1. Introduction:

Dirac put forward the idea of magnetic monopole [1] while the fresh interest on the subject of monopoles was enhanced by the work of t'Hooft [2] and Polyakov [3] and its extension to the case of dyons by Julia and Zee [4]. Consequently, these particles became intrinsic part of all current grand unified theories [5, 6] with their enormous potential importance [7, 8, 9, 10, 11, 12, and 13] in connection with various physical problems. Keeping in view the results of Witten [13] that monopoles are necessarily dyons, a self-consistent and covariant quantum field theory for dyons each carrying the generalized charges as its real and imaginary parts has been constructed [14,15] and accordingly the quaternionic forms of generalized field equations of dyons are developed [2,16,18,18,] in unique, simple and compact notations. On the other hand Kravchenko [20] has analysed the Maxwell equation in the presence of sources, time-dependent electromagnetic fields in homogeneous medium. In our previous papers, we [21, 22] have analysed the generalized Maxwell-Dirac equations in homogeneous medium, developed their quaternionic formulation and also obtained the solutions for the classical problem of moving dyons in simple, compact and consistent manner. Keeping in view all these facts in mind, in this paper, we have studied the Maxwell's equations in presence of electric and magnetic sources (i.e. dyons) and extended the generalized electromagnetic field equations associated with dyons to the case of time-dependent harmonic Maxwell's-Dirac equations of dyon in simple, compact and consistent way. It has been emphasized that the theory reduces to the theory of dynamics of electric (magnetic) charge in the absence of magnetic (electric) charge on dyons

2. Generalized Electromagnetic Fields of Dyons in Homogeneous Medium:

Let us start with the symmetrised Maxwell equations, derived by Dirac [1] in presence of magnetic charge (monopole) to establish the dual invariance between electric and magnetic fields, in the homogeneous media in the following manner [23].for $c = \eta = 1$ i.e.

$$\vec{\nabla} \cdot \vec{D} = \rho_e \qquad (1.a)$$

$$\vec{\nabla} \cdot \vec{B} = \mu_0 \rho_m \qquad (1.b)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\vec{j}_m}{\varepsilon_0} - \frac{\partial \vec{B}}{\partial t} \quad (1.c)$$

$$\vec{\nabla} \times \vec{H} = \vec{j}_e + \frac{\partial \vec{D}}{\partial t} \quad (1.d)$$

where ρ_e and ρ_m are respectively the electric and magnetic charge densities while $\vec{j}_e$ and $\vec{j}_m$ are the corresponding current densities, $\vec{D}$ is electric induction vector, $\vec{E}$ is electric field, $\vec{H}$ is magnetic induction vector and $\vec{B}$ is magnetic field, and ε and μ are defined respectively as permittivity and permeability associated with electric and magnetic fields. Here we assume the homogeneous medium with the following definitions [24],

$$\vec{D} = \varepsilon \vec{E} \quad (\varepsilon = \varepsilon_0 \varepsilon_r) \quad (2)$$

and

$$\vec{B} = \mu \vec{H} \quad (\mu = \mu_0 \mu_r) \quad (3)$$

where ε_0 is the free space permittivity, μ_0 is the permeability of free space, ε_r and μ_r are defined respectively as relative permittivity and permeability associated with electric and magnetic fields. On using equations (2) and (3), equations (1.a - 1.d) take the following differential form [7],

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_e}{\varepsilon} \quad (4.a)$$

$$\vec{\nabla} \cdot \vec{B} = \mu \rho_m \quad (4.b)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\vec{j}_m}{\varepsilon} - \frac{\partial \vec{B}}{\partial t} \quad (4.c)$$

$$\vec{\nabla} \times \vec{B} = \mu \vec{j}_e + \frac{1}{v^2} \frac{\partial \vec{E}}{\partial t} \quad . \quad (4.d)$$

Differential equations (4.a-4.d) are referred as the generalized field equations of dyons in homogeneous medium and the corresponding electric and magnetic fields are then called as the generalized electromagnetic fields of dyons.

The electric and magnetic fields of dyons are expressed in homogeneous medium in terms of two potentials i.e.

$$\vec{E} = -\vec{\nabla} \phi_e - \frac{\partial \vec{C}}{\partial t} - \vec{\nabla} \times \vec{D} \quad (5)$$

$$\vec{B} = -\vec{\nabla} \phi_m - \frac{1}{v^2} \frac{\partial \vec{D}}{\partial t} + \vec{\nabla} \times \vec{C} \quad (6)$$

Where $\{C^\mu\} = \{\phi_e, v\,\vec{C}\}$ and $\{D^\mu\} = \{v\,\phi_m, \vec{D}\}$ are the two four- potentials associated with electric and magnetic charges. Equations (5) and (6) are symmetrically invariant under the following duality transformations;

$$\vec{E} \to v\,\vec{B} \qquad \vec{B} \to -\frac{\vec{E}}{v} \quad (7)$$

$$\vec{C} \to \frac{\vec{D}}{v} \qquad \vec{D} \to -v\,\vec{C} \quad (8)$$

$$\phi_e \to v\,\phi_m \qquad \phi_m \to -\frac{\phi_e}{v} \quad (9)$$

$$\vec{j}_e \rightarrow v\,\vec{j}_m \qquad \vec{j}_m \rightarrow -\frac{\vec{j}_e}{v} \tag{10}$$

$$\rho_e \rightarrow \frac{\rho_m}{v} \qquad \rho_m \rightarrow -v\,\rho_e \tag{11}$$

$$F_{\mu\nu} \rightarrow v\,F^d_{\mu\nu} \qquad F^d_{\mu\nu} \rightarrow -\frac{F_{\mu\nu}}{v}\,. \tag{12}$$

Let us define the complex vector field $\vec{\psi}$ in the following form,

$$\vec{\psi} = \vec{E} - i\,v\,\vec{B} \tag{13}$$

and using equations (5,6) and (13), we get the following relations between generalized field $\vec{\psi}$ and the components of complex four-potential as,

$$\vec{\psi} = -\frac{\partial \vec{V}}{\partial t} - \vec{\nabla}\phi - iv\left(\vec{\nabla} \times \vec{V}\right) \tag{14}$$

Where $\{V_\mu\}$ is the generalized four-potential of dyons in isotropic (homogeneous) medium and defined as,

$$V_\mu = \{\phi, \vec{V}\} \tag{15}$$

i.e.

$$\phi = \phi_e - i\,v\,\phi_m \tag{16}$$

and $$\vec{V} = \vec{C} - i\frac{\vec{D}}{v}\quad . \tag{17}$$

Maxwell field equations (1) may then be written in terms of generalized fields $\vec{\psi}$ as,

$$\vec{\nabla}\cdot\vec{\psi} = \frac{\rho}{\varepsilon} \tag{18}$$

$$\vec{\nabla} \times \vec{\psi} = -i\,v\left(\mu\,\vec{j} + \frac{1}{v^2}\frac{\partial \vec{\psi}}{\partial t}\right) \tag{19}$$

Where ρ and $\vec{j}$ are the generalized charge and current source densities of dyons in homogeneous medium given by [7],

$$\rho = \rho_e - i\frac{\rho_m}{v} \tag{20}$$

$$\vec{j} = \vec{j}_e - iv\vec{j}_m \quad . \tag{21}$$

Taking the curl of equation (19) and using equation (18) we obtain the new parameter (called $\vec{S}$) expressed in the following form in terms of source densities

i.e. $$\vec{S} = \square\vec{\psi} = -\mu\frac{\partial \vec{j}}{\partial t} - \frac{1}{\varepsilon}\vec{\nabla}\rho - iv\mu\left(\vec{\nabla} \times \vec{j}\right) \tag{22}$$

Where $\square$ is the D' Alembertian operator defined as,

$$\square = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \frac{1}{v^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}\,. \tag{23}$$

Maxwell's-Dirac equations (1) may now be expressed in the following manner to establish the relation between generalized potential and current components of dyons i.e.

$$\square\phi = v\mu\rho \tag{24}$$

$$\square\vec{V} = \mu\vec{j} \; . \tag{25}$$

We may thus write the tensorial form of generalized Maxwell's-Dirac equations of dyons in isotropic media [22] as,

$$F_{\mu\upsilon,\upsilon} = j_{\mu}^{e} \tag{26}$$

$$F_{\mu\upsilon,\upsilon}^{d} = j_{\mu}^{m} \; . \tag{27}$$

Defining the generalized field tensor of dyon as,

$$G_{\mu\nu} = F_{\mu\nu} - ivF_{\mu\upsilon}^{d} \; . \tag{28}$$

which yields the following covariant form of generalized field equation of dyon in homogeneous (isotropic) medium i.e.

$$G_{\mu\upsilon,\upsilon} = j_{\mu} \tag{29}$$

$$G_{\mu\upsilon,\upsilon}^{d} = 0 \; . \tag{30}$$

The suitable manifestly covariant Lagrangian density, which yields the fields equations (1) under the variation of field's parameters (i.e. potential) only without changing the trajectory of particle, may be written as follows.

$$L = -m_0 - \frac{1}{4} G_{\mu\upsilon} G_{\mu\upsilon}^{*} + V_{\mu}^{*} j_{\mu} \tag{31}$$

where m_0 is the rest mass of particle and * denotes the complex conjugate.

The Lorentz four-force equation of motion for dyon in homogeneous (isotropic) medium as

$$f_{\mu} = m_0 \ddot{x}_{\mu} = \operatorname{Re} q * \left(G_{\mu\upsilon} u^{\upsilon}\right) \tag{32}$$

where Re denotes the real part , $\ddot{x}_{\mu}$ is the four acceleration and $\{u^{\nu}\}$ is the four velocity of the particle and $q = e - ivg$.

Equations (26), (27) and (29-30) are invariant under duality transformations,

$$\left(F, vF^{d}\right) = \left(F\cos\theta + vF^{d}\sin\theta, -F\sin\theta + vF^{d}\cos\theta\right) \tag{33}$$

$$\left(j_{\mu}, k_{\mu}\right) = \left(j_{\mu}\cos\theta + k_{\mu}\sin\theta, -j_{\mu}\sin\theta + k_{\mu}\cos\theta\right) \tag{34}$$

where the constancy condition is

$$\frac{g}{e} = \frac{B_{\mu}}{A_{\mu}} = \frac{k_{\mu}}{j_{\mu}} = \frac{F^{d}}{F} = -\tan\theta \; . \tag{35}$$

Hence the generalized charge of dyon may be written as

$$q = e - ivg = |q| \exp[-i\theta] . \tag{36}$$

In the homogeneous medium the dual invariant energy density of dyons is now expressed as

$$U = \frac{1}{2}\varepsilon E^{2} + \frac{1}{2\mu} B^{2} \tag{37}$$

3. Quaternion Analyticity of Time-Harmonic Dyonic Field Equation:

Using the Fourier transform any electromagnetic field can be represented as an infinite superposition of time-harmonic (monochromatic) fields.

A time-harmonic electromagnetic field is described as [22, 24]

$$\vec{E}(x,t) = \mathrm{Re}\left(\vec{E}(x)e^{-i\omega t}\right) \tag{38}$$

and

$$\vec{B}(x,t) = \mathrm{Re}\left(\vec{B}(x)e^{-i\omega t}\right) \tag{39}$$

where the electric field $\vec{E}$ and magnetic field $\vec{B}$ depend only on the spatial variables $x = (x_1, x_2, x_3)$ and their true dependence is contained in the factor $e^{-i\omega t}$. $\vec{E}$ and $\vec{B}$ are complex vectors called the complex amplitudes of electromagnetic field and ω is the frequency of oscillations.

Substituting the values of $\vec{E}$ and $\vec{B}$ into the generalized dyonic equations (4.a-4.d), in homogeneous medium, we obtain

$$\begin{aligned}
&\vec{\nabla}\cdot\vec{E} = \frac{\rho_e}{\varepsilon} \\
&\vec{\nabla}\cdot\vec{B} = \mu\rho_m \\
&\vec{\nabla}\times\vec{E} = -i\omega\vec{B} - \frac{\vec{j}_m}{\varepsilon} \\
&\vec{\nabla}\times\vec{B} = -\frac{i\omega}{v^2}\vec{E} + \mu\,\vec{j}_e\,.
\end{aligned} \tag{40}$$

Let us denote $\alpha = \omega\sqrt{\varepsilon\mu} = \frac{\omega}{v}$, where the square root is chosen that $\mathrm{Im}\,\alpha \geq 0$. The quantity α is called the wave number. Let us write the $\vec{D}$, $\vec{E}$ and $\vec{B}$ in the following quaternionic form as [22],

$$D = \partial_1 e_1 + \partial_2 e_2 + \partial_3 e_3 \tag{41}$$

$$E = E_1 e_1 + E_2 e_2 + E_3 e_3 \tag{42}$$

$$B = B_1 e_1 + B_2 e_2 + B_3 e_3 \tag{43}$$

where e_1, e_2 and e_3 are the elements of a quaternion and satisfy the following multiplication rule,

$$\begin{aligned}
&e^2{}_0 = 1 \\
&e_j e_k = -\delta_{jk} e_0 + \varepsilon_{jkl} e_l
\end{aligned} \tag{44}$$

where ε_{jkl} $(j,k,l = 1,2,3 \; and \; e_0 = 1)$ is Levi - Civita three index symbols and δ_{jk} is Kronecker delta. Using equations (41-43), we get the following quaternionic differential equations i.e.

$$\begin{aligned}
D\vec{E} &= (\partial_1 e_1 + \partial_2 e_2 + \partial_3 e_3)(E_1 e_1 + E_2 e_2 + E_3 e_3) \\
&= -\frac{\rho_e}{\varepsilon} - \frac{\vec{j}_m}{\varepsilon} + i\,\omega\vec{B}
\end{aligned} \tag{45}$$

and
$$D\vec{B}=(\partial_1 e_1+\partial_2 e_2+\partial_3 e_3)(B_1 e_1+B_2 e_2+B_3 e_3) \\ =-\mu\rho_m+\mu\vec{j}_e-i\frac{\omega}{v^2}\vec{E} \quad . \tag{46}$$

Let us introduce the following pairs of purely vectorial biquaternionic functions

i.e.
$$\vec{l}=-\frac{i\omega}{v^2}\vec{E}+\alpha\vec{B} \tag{47}$$

and
$$\vec{m}=\frac{i\omega}{v^2}\vec{E}+\alpha\vec{B}. \tag{48}$$

Taking the divergence of third and fourth equation (40), we get the following pairs of continuity equation for electric and magnetic charges i.e.

$$\vec{\nabla}\cdot\vec{j}_e-i\omega\rho_e=0 \tag{49}$$

and
$$\vec{\nabla}\cdot\vec{j}_m-i\omega\varepsilon\mu\rho_m=0. \tag{50}$$

Applying the quaternionic operator D given by equation (41) to the quaternionic form of $\vec{l}$ and using equations (45- 46) and (49 - 50), we get

$$D\vec{l}=\frac{1}{\varepsilon v^2}\left[\vec{\nabla}\cdot\vec{j}^*\right]+\alpha\mu\vec{j}^*+\alpha\vec{l} \tag{51}$$

where $\vec{j}^*$ is the complex conjugate of dyonic current density in homogenous medium given by equation (21). Thus $\vec{l}$ satisfied the equation which is derived by equation (51) as,

$$(D-\alpha)\vec{l}=\mu\left[\vec{\nabla}\cdot\vec{j}^*\right]+\alpha\mu\vec{j}^* \tag{52}$$

Analogous to equation (52), $\vec{m}$ also satisfies the equation (52) as,

$$(D+\alpha)\vec{m}=-\mu\left[\vec{\nabla}\cdot\vec{j}\right]+\alpha\mu\vec{j}. \tag{53}$$

Thus, the process of diagonalization can be written in the matrix form as

$$\begin{pmatrix} D & -i\omega \\ i\omega & D \end{pmatrix}\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix}=B_\alpha\begin{pmatrix} D-\alpha & 0 \\ 0 & D+\alpha \end{pmatrix}B_\alpha^{-1}\begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \tag{54}$$

where
$$B_\alpha=\begin{pmatrix} \frac{-i\omega}{v^2} & \alpha \\ \frac{i\omega}{v^2} & \alpha \end{pmatrix} \tag{55}$$

and
$$B_\alpha^{-1}=\begin{pmatrix} \frac{-v^2}{i\omega} & \frac{v^2}{i\omega} \\ \frac{1}{\alpha} & \frac{1}{\alpha} \end{pmatrix}. \tag{56}$$

As such, we have obtained the two decoupled equations for the unknown vectors $\vec{l}$ and $\vec{m}$ which simplifies the analysis of the generalized Dirac-Maxwell (GDM) equation of dyons in homogeneous (isotropic) medium. These equations reduce to the theory of electric (magnetic) charges predicted earlier by Kravchenko [20] in the absence of magnetic (electric) charge or vice-versa.

4. Conclusion:

Equations given by (4.a-4.d) are described as the generalized Dirac-Maxwell (GDM) equations in presence of electric and magnetic charge i.e. dyons. The constitutive relations (relation between induction and field vector) given by $\vec{D} = \varepsilon \vec{E}$ $(\varepsilon = \varepsilon_0 \varepsilon_r)$ and $\vec{B} = \mu \vec{H}$ $(\mu = \mu_0 \mu_r)$ describe the rich variety of physical phenomenon representing the properties and response of the medium and to the application of generalized electromagnetic field of dyons. The generalized Dirac - Maxwell's (GDM) equation, generalized field tensors, Lagrangian density, the equation of motion (Lorentz four force) and energy density of dyons in homogeneous medium remains invariant under Lorentz and duality transformations. The field equations, Lagrangian density and the equation of motion of dyons describe here in homogeneous medium are also described as Poincare and conformal invariant, but there is no trivial gauge group of invariance transformations. The constancy condition relates the electric and magnetic constituents to avoid the existence of other photons. As such theory of dyons does not describe the existence of two photons if we incorporate the constancy condition. Here we have described characteristics of dyonic field equations in terms of the time independent parameters of the medium ε and μ. Thus we have represented the generalized electromagnetic fields of dyons in terms of time-harmonic fields in a consistent manner and reproduce the theories of the dynamics of electric (magnetic) charge in the absence of magnetic (electric) charge on dyon or vice-versa. We have used the Fourier transformation where any electromagnetic field is represented as an infinite superposition of time harmonic (monochromatic) fields. These fields are normally the main objects of study in radio electronics, wave propagation theory and many other branches of physics and engineering. For the unknown vectors $\vec{l}$ and $\vec{m}$ (instead of electric and magnetic fields), we have obtained the two decoupled equations which simplifies the analysis of the generalized Dirac-Maxwell equation of dyons in homogeneous medium after the diagonalization process. These equations reduce to the theory of electric (magnetic) charge predicted earlier by Kravchenko [20] in the absence of magnetic (electric) charge on dyon or vice-versa.

5. References:

[1] Dirac P. A .M, Proc. R. Soc. London, **A133** (1931), 60.
[2] t' Hooft G., Nucl. Phys., **B79** (1974),276.
[3] Polyakov A. M., JEPT Lett, **20** (1974), 194.
[4] Julia B. and Zee A., Phys. Rev., **D11** (1975), 2227.
[5] Dokos C. and Tomaros T., Phy. Rev., **D21** (1980), 2940.
[6] Preskill J. Annu. Rev. Nucl. Part. Sci., **34** (1984), 461.
[7] Callen C. G., Phys. Rev. **D25** (1982), 2141.
[8] Rubakov V. A., Nucl. Phy., **B203** (1982), 211.
[9] Mandelstam, S. Phys. Rev., **D 19** (1976), 249.
[10] t'Hooft G., Nucl. Phys., **B138** (1978), 1.
[11] Rajput, B. S., J. Math. Phys., **25** (1984), 351.

[12] Rajput ,B. S. and Rashami Gunwant, Ind. J. Pure and Appl. Phys., **26** [1988], 538.
[13] Witten E., Phy. Lett. B86 (1979), 283.
[14] Rajput, B. S. and Joshi D.C., Had. J., 4 (1981), 1805.
[15] Rajput, B. S. and Bhakuni, D.S., Lett. Nuovo Cimento, **34** (1982), 509.
[16] Bisht P. S., Negi, O. P. S and Rajput B. S., Int. J. Theor. Phys., **32** (1993) 2099.
[17] Bisht, P. S., Negi, O. P. S. and Rajput, B. S., IL Nuovo Cimento, **104A**(1991), 337.
[18] Bisht P. S., Negi O. P. S and Rajput B. S., Prog. Theor. Phys., **85** (1991) 151.
[19] Bisht Shalini, Bisht P. S. and Negi O. P. S. , Nuovo Cimento, **B113,** (1998) 1449.
[20] Kravchenko, V.V, Applied Quaternionic Analysis, Research and Exposition in Mathematics , **28** (2003), Heldermann Verlage, Germany.
[21] Singh. Jivan, Bisht, P. S. and Negi, O. P. S , "Generalized fields of dyon in Isotropic medium" arXiv: hep-th/0611208; Communication in Physics, Vol. **17**, No.2 (2007), pp.83-90
[22] Singh. Jivan, Bisht, P. S. and Negi, O. P. S, "Quaternion analysis of Generalized fields of dyons in isotropic medium." arXiv: hep-th/07033083; J. Phys. A: Math. Theor. **40** (2007) 9137-9147.
[23] Barker W. A. and Graziani Frank, Am. J. Phys., **46** (1978) 1111.
[24] Starton J. A., Electromagnetic Theory, Mc Graw Hill Company, New York, (1941).

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